

# Novel Explicit Multi Spin String Solitons in $AdS_5$

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## Abstract

We find new explicit solutions describing closed strings spinning with equal angular momentum in two independent planes in  $AdS_5$ . These are  $2N$ -folded strings in the radial direction and also winding  $M$  times around an angular direction. Thus in spacetime they consist of  $2N$  segments. Solutions fulfilling the closed string periodicity conditions exist provided  $N/M > 2$ , i.e. the strings must be folded at least six times in the radial coordinate. The strings are spinning, or actually orbiting, similarly to solutions found previously in black hole spacetimes, but unlike the one-spin solutions in  $AdS$  which spin around their center. For long strings we recover the logarithmic scaling relation between energy and spin known from the one-spin case, but different from other known two-spin cases.

# 1 Introduction and Results

In connection with the conjectured duality [1, 2, 3] between super string theory on  $AdS_5 \times S^5$  and  $\mathcal{N} = 4$  SU(N) super Yang-Mills theory in Minkowski space, there has been a lot of interest in string solitons in  $AdS_5$ ,  $S^5$ ,  $AdS_5 \times S^5$  and other related backgrounds. Rigidly rotating strings in  $AdS$  [4] (as well as circular pulsating strings [5]) were considered long time ago, but the direct connection with gauge theory was only discovered quite recently [6]. Namely, it was noticed that the subleading term in the scaling relation between energy  $E$  and spin  $S$ , for the rigidly rotating strings, is logarithmic

$$E - S \sim \ln(S) \tag{1.1}$$

similarly to results found for certain Yang-Mills operators back in the early days of QCD [7, 8, 9, 10, 11]. The paper [6] initiated a whole industry of finding new string solitons, and in some cases also the corresponding Yang-Mills operators. See for instance [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26] for various string configurations.

More recently it was discovered that multi spin solutions could also be constructed easily [27]. Among others, a relatively simple solution was found in  $AdS_5$  describing a string which is located at a point in the radial direction, winding around an angular direction and spinning with equal angular momentum in two independent planes. In the long string limit it was shown that

$$E - 2S \sim S^{1/3} \tag{1.2}$$

differing substantially from (1.1). This is not at all a problem though, since the subleading terms are well-known to depend on the particular string configuration. Multi spin solitons have been further investigated in a number of papers including [26, 28, 29, 30, 31, 32, 33, 34, 35, 36] (for a review see [37]). In particular, in [32] it was shown that the solutions to the string equations of motion can be classified and related to solutions of the integrable Neumann system. However, multi spin solutions were mostly considered in  $S^5$  in the abovementioned papers. Even though  $AdS_5$  and  $S^5$  geometrically only differ by a couple of signs, the scaling relations between energy and spin for string solitons are completely different. So, much more work needs to be done for multi spin solutions in  $AdS_5$ .

While many explicit multi spin solutions in  $S^5$  have been constructed [27, 30, 31, 32, 33, 34, 37], it seems that there are essentially still only two

explicit multi spin solutions known in  $AdS_5$ , namely the one-parameter family [27] leading to (1.2) and its generalisation to arbitrary unequal angular momenta [34, 37], which gives a somewhat similar (although non-regular) scaling relation.

In the present paper we construct another one-parameter family of multi spin solutions in  $AdS_5$ . Our solutions describe  $2N$ -folded strings in the radial direction and also winding  $M$  times around an angular direction. They are spinning with equal angular momentum in two independent planes. For some reason, for which we have no physical explanation, the closed string periodicity conditions can only be fulfilled provided  $N/M > 2$ , i.e., the strings must be folded at least six times in the radial coordinate. But it must be stressed that the strings are not folded in spacetime; they consist of  $2N$  segments.

Moreover, our strings are quite different from the usual straight strings spinning in  $AdS$  [4], since they are not spinning around their center. Instead, the whole string is spinning, or actually orbiting, around origo similarly to solutions previously found in black hole spacetimes [4, 15, 38, 39]. There is of course no central attractive body in this case; instead, the orbits are stabilised by the string winding around an angular direction.

For long strings we find that

$$E - 2S \sim \ln(S) \tag{1.3}$$

similarly to (1.1) except for the obvious factor 2, but completely different from (1.2). Thus we recover the remarkable logarithmic scaling relation from the one-spin case [6]. It must be stressed that our solutions are "orthogonal" to the solutions of [27] in the sense that there is only one common solution in the two families, and this particular solution is not a long string.

The paper is organised as follows. In Section 2, we present our ansatz and solve the resulting equations of motion explicitly. We also obtain the periodicity conditions in compact form. In Section 3, we solve the periodicity conditions for short and long strings, and analyse the corresponding string configurations. We also comment on the relation to previously known results. In Section 4, we compute the energy  $E$  and the two spins  $S_1 = S_2 \equiv S$ , and derive the logarithmic scaling relation (1.3) in the limit of long strings. Finally in Section 5, we present our conclusions as well as some comments about stability.

## 2 Two-Spin Solutions in $AdS_5$

We take the  $AdS_5$  line-element in the form

$$ds^2 = -(1 + H^2 r^2) dt^2 + \frac{dr^2}{1 + H^2 r^2} + r^2 (d\beta^2 + \sin^2 \beta d\phi^2 + \cos^2 \beta d\tilde{\phi}^2) \quad (2.1)$$

where  $H^{-1}$  is the scale of  $AdS_5$ . The 't Hooft coupling in this notation is  $\lambda = (H^2 \alpha')^{-2}$ .

The string which is extended in the  $r$  and  $\beta$  directions and spinning with equal spin in the  $\phi$  and  $\tilde{\phi}$  directions, is obtained by the ansatz

$$t = c_0 \tau, \quad r = r(\sigma), \quad \beta = \beta(\sigma), \quad \phi = \omega \tau, \quad \tilde{\phi} = \omega \tau \quad (2.2)$$

where  $c_0$  and  $\omega$  are arbitrary constants. Then the  $r$  and  $\beta$  equations become

$$r'' - \frac{r'^2 H^2 r}{1 + H^2 r^2} - \beta'^2 (1 + H^2 r^2) r - H^2 c_0^2 (1 + H^2 r^2) r + \omega^2 (1 + H^2 r^2) r = 0 \quad (2.3)$$

$$\beta'' + \frac{2r'\beta'}{r} = 0 \quad (2.4)$$

while the non-trivial conformal gauge constraint is

$$\frac{r'^2}{1 + H^2 r^2} + r^2 \beta'^2 - c_0^2 (1 + H^2 r^2) + r^2 \omega^2 = 0 \quad (2.5)$$

The system is reduced to first order form

$$\beta' = \frac{k}{r^2} \quad (2.6)$$

$$\frac{r'^2}{1 + H^2 r^2} + \frac{k^2}{r^2} + r^2 \omega^2 - c_0^2 (1 + H^2 r^2) = 0 \quad (2.7)$$

where  $k$  is an integration constant. These equations are identical to those considered in [27] (in the parametrisation  $Hr = \sinh \rho$ ), but only constant  $r$  solutions were found there. Similar equations and their generalisations were also considered in [34, 37], but again only constant  $r$  solutions were explicitly constructed. We shall now show that solutions with  $r = r(\sigma)$  exist as well, provided that the string is folded at least six times in the radial direction.

First concentrate on the  $r$  equation (2.7). In the non-negative dimensionless coordinate  $y \equiv H^2 r^2$ , it can be written

$$y'^2 = 4(H^2 c_0^2 - \omega^2)y^3 + 4(2H^2 c_0^2 - \omega^2)y^2 + 4(H^2 c_0^2 - H^4 k^2)y - 4H^4 k^2 \quad (2.8)$$

If  $H^2 c_0^2 - \omega^2 = 0$ , the solution is

$$y = \frac{-(c_0^2 - H^2 k^2)}{2c_0^2} \pm \frac{(c_0^2 + H^2 k^2)}{2c_0^2} \cosh(2H c_0 \sigma) \quad (2.9)$$

but it must be discarded since it is not periodic. If  $H^2 c_0^2 - \omega^2 \neq 0$ , we notice that the right hand side can be factorized

$$\begin{aligned} y'^2 = & 4(H^2 c_0^2 - \omega^2)(y + 1) \\ & \left( y - \left( \frac{H^2(-c_0^2 + \sqrt{c_0^4 + 4k^2(H^2 c_0^2 - \omega^2)})}{2(H^2 c_0^2 - \omega^2)} \right) \right) \\ & \left( y - \left( \frac{H^2(-c_0^2 - \sqrt{c_0^4 + 4k^2(H^2 c_0^2 - \omega^2)})}{2(H^2 c_0^2 - \omega^2)} \right) \right) \end{aligned} \quad (2.10)$$

To get periodic solutions, we are interested in the case with two non-negative roots, so we get the conditions

$$H^2 c_0^2 - \omega^2 < 0 \quad (2.11)$$

$$c_0^4 + 4k^2(H^2 c_0^2 - \omega^2) \geq 0 \quad (2.12)$$

Now define the function  $P$

$$y = \frac{P}{H^2 c_0^2 - \omega^2} - \frac{2H^2 c_0^2 - \omega^2}{3(H^2 c_0^2 - \omega^2)} \quad (2.13)$$

such that

$$P'^2 = 4P^3 - g_2 P - g_3 = 4(P - e_1)(P - e_2)(P - e_3) \quad (2.14)$$

where the roots ( $e_1 > e_2 \geq e_3$ ) are

$$e_1 = \frac{1}{3}(2\omega^2 - H^2 c_0^2) \quad (2.15)$$

$$e_2 = \frac{1}{6}H^2c_0^2 - \frac{1}{3}\omega^2 + \frac{1}{2}H^2\sqrt{c_0^4 + 4k^2(H^2c_0^2 - \omega^2)} \quad (2.16)$$

$$e_3 = \frac{1}{6}H^2c_0^2 - \frac{1}{3}\omega^2 - \frac{1}{2}H^2\sqrt{c_0^4 + 4k^2(H^2c_0^2 - \omega^2)} \quad (2.17)$$

and the invariants are  $g_2 = 2(e_1^2 + e_2^2 + e_3^2)$  and  $g_3 = 4e_1e_2e_3$ . Notice also that  $\Delta \equiv g_2^3 - 27g_3^2 \geq 0$  (using (2.11), (2.12)).

Equation (2.14) is the Weierstrass equation with solution

$$P(\sigma) = \wp(\sigma + a) \quad (2.18)$$

where  $\wp$  is the doubly-periodic Weierstrass function and  $a$  is a complex integration constant [40]. To get a real non-singular solution, we take  $a$  to be half the imaginary period. Then the Weierstrass function reduces to a Jacobi elliptic function [40]. Before writing down the explicit solution, it is convenient to trade the parameters  $(c_0, \omega, k)$  for

$$b = e_2 - e_3 = H^2\sqrt{c_0^4 + 4k^2(H^2c_0^2 - \omega^2)} \quad (2.19)$$

$$m = \frac{e_2 - e_3}{e_1 - e_3} = \frac{2H^2\sqrt{c_0^4 + 4k^2(H^2c_0^2 - \omega^2)}}{2\omega^2 - H^2c_0^2 + H^2\sqrt{c_0^4 + 4k^2(H^2c_0^2 - \omega^2)}} \quad (2.20)$$

$$n = \frac{2\sqrt{c_0^4 + 4k^2(H^2c_0^2 - \omega^2)}}{c_0^2 + \sqrt{c_0^4 + 4k^2(H^2c_0^2 - \omega^2)}} \quad (2.21)$$

such that by eqs.(2.11)-(2.12)

$$b \geq 0, \quad 0 \leq m \leq n \leq 1 \quad (2.22)$$

In this parametrisation the solution to (2.7) is

$$H^2r^2(\sigma) = \frac{m}{n-m} \left( 1 - n \operatorname{sn}^2 \left( \sqrt{\frac{b}{m}} \sigma | m \right) \right) \quad (2.23)$$

so that  $Hr(\sigma)$  oscillates between the non-negative values  $Hr_{max}$  and  $Hr_{min}$ , where

$$Hr_{max} = \sqrt{\frac{m}{n-m}}, \quad Hr_{min} = \sqrt{\frac{m(1-n)}{n-m}} \quad (2.24)$$

This means that the whole string is spinning, or actually orbiting, around origo. It is quite similar to straight strings spinning around black holes [4, 15, 38, 39], but unlike one-spin strings in *AdS* which spin around their center [4].

The solution (2.23) must be supplemented with the closed string periodicity condition  $r(\sigma + 2\pi) = r(\sigma)$ . For a  $2N$ -folded closed string ( $N$  positive integer), this condition becomes

$$\frac{\sqrt{m}K(m)}{\pi\sqrt{b}} = \frac{1}{N} \quad (2.25)$$

which trivially gives  $b$  in terms of  $m$ . This condition of course drops out for  $n = 0$  (which implies  $b = m = 0$ ), corresponding to strings which are pointlike in the radial direction [27]. Here we shall only consider strings which are extended in the radial direction.

Now return to the  $\beta$  equation (2.6). It is trivially integrated to

$$\begin{aligned} \beta(\sigma) &= \sqrt{\frac{b(n-m)(1-n)}{nm}} \int_0^\sigma \frac{dx}{1 - n \operatorname{sn}^2\left(\sqrt{\frac{b}{m}}x|m\right)} \\ &= \sqrt{\frac{(n-m)(1-n)}{n}} \Pi\left(n; \sqrt{\frac{b}{m}}\sigma|m\right) \end{aligned} \quad (2.26)$$

Since  $\beta$  is an angular coordinate, we impose the quasi-periodicity condition  $\beta(\sigma + 2\pi) = 2M\pi + \beta(\sigma)$ , where  $M$  is another positive integer which plays the role of a winding number. This condition translates into

$$\frac{\sqrt{(1-n)(n-m)}\Pi(n;m)}{\pi\sqrt{n}} = \frac{M}{N} \quad (2.27)$$

where (2.25) was also used. This equation determines  $n$  in terms of  $m$ , thus the only free parameter (for fixed  $N, M$ ) now is  $m$ .

### 3 Analysis of the Solutions

The main problem when analysing the solutions of the previous section, is that (2.27) is transcendental. Numerical analysis shows that the left hand side is less than  $1/2$  (it equals  $1/2$  only for  $m = 0$ , corresponding to strings



which are pointlike at  $r = 0$ ). Thus, to get non-trivial solutions, we need  $N > 2M$  which means that the string must be folded at least six times. In that case we get a one-parameter family of solutions with arbitrary extension in the radial direction.

For short and long strings, we can solve (2.27) analytically and confirm the numerical results. First consider short strings. From (2.23) follows that short strings correspond to  $m \approx 0$ ,  $n \approx 0$  (with  $m \leq n$ ). Using that  $\Pi(n; m) \approx \pi/2$  in this case, we get from (2.27)

$$\frac{\sqrt{n-m}}{2\sqrt{n}} \approx \frac{M}{N} \quad (3.1)$$

such that

$$\frac{n}{m} \approx \frac{N^2}{N^2 - 4M^2}, \quad \text{short strings} \quad (3.2)$$

which implies  $N > 2M$ . For such short strings, in the limit where  $r$  is almost a constant, we get

$$Hr \approx \sqrt{\frac{N^2 - 4M^2}{4M^2}} \quad (3.3)$$

$$\beta \approx M\sigma \quad (3.4)$$

thus the string is located at a fixed finite value of  $r$  and simply winding around in the  $\beta$  direction. This is a special case of the solutions found in [27]. To get all the solutions of [27], one should discard the periodicity condition, as discussed after (2.25). But notice that our notion of short means short in the radial extension, which is different from the notion of short used elsewhere.

Now consider long strings, which correspond to  $n \approx 1$ ,  $m \approx 1$  (with  $m \leq n$ ). Here we use the approximation

$$\Pi(n; m) \approx \frac{\pi}{2} \sqrt{\frac{1}{(1-n)(n-m)}} \left( 1 - \frac{2}{\pi} \sin^{-1} \sqrt{\frac{1-n}{1-m}} \right) \quad (3.5)$$

Then (2.27) gives

$$\frac{M}{N} \approx \frac{1}{2} - \frac{1}{\pi} \sin^{-1} \sqrt{\frac{1-n}{1-m}} \quad (3.6)$$

such that

$$\sqrt{\frac{1-n}{1-m}} \approx \cos \left( \frac{M}{N} \pi \right), \quad \text{long strings} \quad (3.7)$$

which again implies  $N > 2M$ . This leads to the following expression for  $Hr_{min}$

$$Hr_{min} \approx \cotan\left(\frac{M}{N}\pi\right) \quad (3.8)$$

Thus the long strings extend outwards from this fixed value. We observe that the long strings come closer to origo than the short strings.

## 4 Energy and Spin

The conserved energy is given by

$$E = \frac{c_0}{2\pi\alpha'} \int_0^{2\pi} (1 + H^2 r^2(\sigma)) d\sigma \quad (4.1)$$

In our parametrisation, this can be expressed in terms of a complete elliptic integral as follows

$$E = \frac{N\sqrt{(2-n)nm}}{\pi H\alpha'(n-m)} E(m) \quad (4.2)$$

Similarly, the conserved spins are

$$S_1 = \frac{\omega}{2\pi\alpha'} \int_0^{2\pi} r^2(\sigma) \cos^2 \beta(\sigma) d\sigma, \quad S_2 = \frac{\omega}{2\pi\alpha'} \int_0^{2\pi} r^2(\sigma) \sin^2 \beta(\sigma) d\sigma \quad (4.3)$$

such that [32]

$$\frac{E}{H^2 c_0} - \frac{S_1}{\omega} - \frac{S_2}{\omega} = \frac{1}{H^2 \alpha'} \quad (4.4)$$

In our case,  $S_1 = S_2 \equiv S = (S_1 + S_2)/2$ , which in terms of complete elliptic integrals gives

$$S = \frac{N\sqrt{n+m-nm}}{2\pi H^2 \alpha'(n-m)\sqrt{n}} ((m-n)K(m) + nE(m)) \quad (4.5)$$

Further simplifications are obtained for short and long strings, respectively. For short strings, using (3.2), we arrive at the following finite expressions for

the energy and spin

$$E \approx \frac{N^2 \sqrt{N^2 - 4M^2}}{4\sqrt{2}H\alpha' M^2} \quad (4.6)$$

$$S \approx \frac{(N^2 - 4M^2) \sqrt{N^2 - 2M^2}}{8\sqrt{2}\alpha' H^2 M^2} \quad (4.7)$$

But recall that these are just the minimal values within our family of strings. As explained after (3.4), these strings are not short in the usual sense. In fact, we don't have solutions with  $E \approx 0$  and  $S \approx 0$  within our family of strings.

For the long strings we can eliminate  $n$  using eq.(3.7) and get

$$E \approx \frac{N}{\pi H \alpha' \sin^2 \left( \frac{M}{N} \pi \right) (1-m)} \left( 1 + \frac{1-m}{4} \log \frac{16}{1-m} \right) \quad (4.8)$$

$$S \approx \frac{N}{2\pi \alpha' H^2 \sin^2 \left( \frac{M}{N} \pi \right) (1-m)} \left( 1 - \left( 2 \sin^2 \left( \frac{M}{N} \pi \right) - 1 \right) \frac{1-m}{4} \log \frac{16}{1-m} \right) \quad (4.9)$$

which leads to

$$E/H - 2S \approx \frac{N}{2\pi H^2 \alpha'} \log(2\pi H^2 \alpha' S) \quad (4.10)$$

up to an unimportant additive constant.

Thus, we recover the remarkable and celebrated logarithmic scaling relation known from the one-spin case [6]. This should be contrasted with the two-spin solutions considered in [27], where it was found that (in our notation)

$$E/H - 2S \approx \frac{3(H^2 \alpha' S)^{1/3}}{2^{4/3} H^2 \alpha'} \quad (4.11)$$

## 5 Concluding Remarks

In conclusion, we have constructed a new one-parameter family of multi spin string solitons in  $AdS_5$ . Contrary to the well-known one-spin solutions in  $AdS$  [4], our solutions are orbiting around origo. More importantly, contrary to other explicitly known multi spin solutions [27, 34], our solutions give the logarithmic scaling relation between energy and total spin  $E - 2S \sim \ln(S)$ . This suggests that the corresponding Yang-Mills operator can be constructed along the lines of [6].

Our solutions can be generalised straightforwardly in various ways to solutions in  $AdS_5 \times S^5$ . Either by adding one or more R-charges [27] or by adding pulsation [26].

In order to test the stability of our solutions, it would be interesting to consider linearised perturbations around them. Unfortunately, it turns out to be extremely complicated because of the highly non-trivial sigma-dependent  $r(\sigma)$  and  $\beta(\sigma)$ . As a result, we will get sigma-dependent coefficients in the 3 coupled equations for the physical perturbations, which should be compared with the case considered in [27] where the coefficients were constant. At the moment, we have no solution to this problem. However, we can test our solutions in the limit where they reduce to a solution of [27]. Namely our short strings (which are not short in the sense of [27]), where the spin is given by (4.7). If we take the simplest allowed case ( $N = 3$ ,  $M = 1$ ), we get  $S = 5\sqrt{7\lambda}/2/8 \approx 1.17\sqrt{\lambda}$ , which is far below the set-in of instability found in [27], thus ensuring us of a stable solution.

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